Classification criterion for dynamical systems in intermittent chaos

Zoltán Toroczkai and Áron Péntek
Physics Faculty, Babes-Bolyai University of Cluj, Kogalniceanu 1, 3400-Cluj, Romania
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Intermittent chaos is investigated by means of an extended version of the statistical-mechanical formalism developed by Sato and Honda [Phys. Rev. A **42**, 3233 (1990)]. An exact criterion is given to classify intermittent systems from the point of view of the generated chaotic phases based on the probability distribution of laminar lengths which is an explicitly measurable quantity from the time series. This criterion provides us with the generalization of the concept of intermittency which broadens the class of critical phenomena associated with the spectrum of dynamical entropies. It is shown that, in contrast to general belief, the presence of the regular chaos phase (i.e., vanishing Rényi entropies for inverse temperatures $q > 1$) is not necessarily related to intermittency. In fact, the absence of any phase transition or the appearance of an anomalous chaos phase (i.e., infinite Rényi entropies for $q < 0$) is also possible in intermittent systems. We derive how the pressure, computed from a series of signals of increasing length, approaches its asymptotic value in the regular and anomalous phases.

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I. INTRODUCTION AND SUMMARY

Intermittent chaos is a very often observed phenomenon in dynamical systems [1–3]. Recently, a new, successful statistical-mechanical formalism has been developed by Sato and Honda [4] (referred to as SH hereafter) in order to describe certain features of intermittency by using an analogy between intermittent systems and one-dimensional (1D) lattice-gas models [4–7]. They have also investigated the problem of phase transitions associated with the singularities in the spectrum of generalized entropies and obtained various critical exponents from the q-weighted (with q the inverse temperature) power spectrum, order parameter, and free energy [5].

It is known that, according to qualitatively different dynamical behaviors, several chaotic phases may exist in the spectrum of the Rényi entropies [8], $K(q)$, of chaotic dynamical systems as pointed out by Csordás and Szépfalusy [9]: (i) the chaotic chaos phase (CCP) corresponding to a region in the spectrum of the entropies where $K(q)$ is nonzero and finite, (ii) the regular chaos phase (RCP) where $K(q)$ is zero, and (iii) the anomalous chaos phase (ACP), which is characterized by infinite Rényi entropies (see also Ref. [10]).

It is interesting to study in which chaotic dynamical system one or another of these chaotic phases arises. In the present paper we give an exact answer to this question for intermittent systems in the light of the formalism introduced by SH, providing us with the generalization of the notion of intermittency. As widely believed, intermittency occurs when long series of coherent laminar behaviors can be observed. In the following we use the term *intermittent* for chaotic dynamical systems which spend much more time in a quasiregular or laminar region of the phase space than in the chaotic one. As it is well known, three types of intermittency have been distinguished by Pomeau and Manneville [1,11,12] according to the ways a fixed point loses its stability. This classification, however, tells nothing about the phase-transition phenomena associated with the multifractal properties of the dynamics. In the past few years considerable efforts have been made to elucidate the multifractal properties of chaos; it seems worth working out a classification from the point of view of dynamical phase transitions. Using the SH formalism seems to be a very appropiate way to reach this goal.

In experimentally obtained intermittent signals the most essential, and simultaneously a simply measurable, quantity is the probability distribution of laminar lengths (PDLL), which is also one of the most important inputs of the SH theory. Our aim here is to formulate a criterion for phase transitions in the $K(q)$ spectrum using the PDLL of the measured signal or, more generally, on the basis of the probabilities of homogeneous blocks of symbols labeling different states of the system.

Systems are generally considered to be intermittent if they produce signals with long laminar sequences and, simultaneously, possess asymptotic power-law decay in their time correlation. This behavior always yields the appearance of an RCP phase, i.e., $K(q) = 0$ for $q > 1$. In this paper we apply a somewhat more flexible definition of intermittency, and consider a signal being intermittent whenever it *looks* intermittent, i.e., whenever long uniform blocks show up frequently in the symbolic codes of the dynamics independently of the form of the correlation decay. In other words, our requirement is that the PDLL takes on relatively large values for short and intermediate lengths. In particular, it can happen that in this region the PDLL is not monotonically decreasing, but has some structure, e.g., a local maximum at some finite length. If the PDLL is not negligible on short and intermediate lengths, we consider the system intermittent anyway, even if the asymptotic form of the PDLL or of the correlation decay is not of power-law type. Consequently, intermittency of our sense need not be connected with any phase transition in $K(q)$, or it may also happen that a transition different from the CCP-RCP one occurs. In fact, we show that a CCP-ACP transition can also be observed in intermittent signals.

As it will be seen in the following sections, nonanalyticies of the dynamical multifractal spectrum corresponding to phase transitions in the thermodynamical formalism are determined by the *asymptotic* decay rate of the probability to find long homogeneous symbolic blocks. We show that the existence of a CCP-RCP or CCP-ACP phase transition *uniquely* follows from the asymptotic properties of the PDLL above. Furthermore, the approach to the exact results $K(q) = 0$ or $K(q) = \infty$ can also be expressed by means of PDLL (see Sec. IV).

We shall use an extended version of the Sato-Honda formalism, which is more suitable for our investigations (Sec. II C). We believe that this way of extending the SH model can be continued to build up a formalism valid for the general chaotic case. A discussion in this direction with hints and expectations is given in Sec. VIII.

The paper is organized as follows. In Sec. II we give a brief review of the SH formalism restricted to a binary symbolic dynamics as well as the above-mentioned extension of the same formalism. Section III includes the central result of the paper formulating the criterion for the existence of phase transitions between the phases introduced. In Sec. IV we derive the scaling toward the exact values for the Rényi entropies $K(q) \equiv 0$ or $K(q) \equiv \infty$ in the regular chaos phase and anomalous chaos phase, respectively. In Secs. V, VI, and VII we investigate the situations when PDLL decays slower than exponentially, exponentially, or faster than exponentially, respectively, with examples for the corresponding new types of intermittencies. Section VIII is devoted to an outlook and summary.

II. MULTIFRACTAL FORMALISM FOR INTERMITTENCY

A. Encoding and multifractal spectra of the dynamics

Here we give a brief summary of the multifractal theory relevant for the present study [4,13–15]. Let us consider an intermittent signal. The encoding of this signal can be performed by using two finite symbol sets $\mathcal{E}_1$ and $\mathcal{E}_2$ being associated with the laminar and the chaotic states, respectively. The choice of these symbols depends on particular features of the system. In the following we use the simplest symbolic dynamics which contains just two symbols 0 and 1 corresponding to a partition of the phase space in two distinct phases: phase 0 and phase 1. [Note that the word "phases" has to be understood here as indicating different regions in the phase space, in contrast to the names RCP, CCP, and ACP where it means q phases, i.e., certain parts of the $K(q)$ spectrum.] Symbol 0 always denotes a laminar state $0 \in \mathcal{E}_1$, but symbol 1 could belong to either $\mathcal{E}_2$ or $\mathcal{E}_1$ denoting a chaotic or another laminar state (see Sec. V). The use of this simple encoding is motivated by the fact that it turns out to be sufficient to characterize type-II and -III intermittency [11,12]. Note, however, that to describe type-I intermittency at least three (two laminar and one chaotic) symbols are needed [7]. In what follows we use the simple symbolic dynamics in order to present the main features of the multifractal theory. The case of type-I intermittency will be discussed in Sec. VIII.

For intermittent systems a nontrivial natural measure need not exist because of the dominating laminar behavior. In the following we consider strictly chaotic intermittent systems with positive averaged Lyapunov exponent and a nontrivial natural measure ensuring, therefore, that the probabilities defined below exist. Let us take a partition function

$$Z_n(q) = \sum_{\{s_i\}_n} [p(\{s_i\}_n)]^q, \tag{1}$$

where s_i is either 0 or 1 assigned to either phase 0 or phase 1, and $p(\{s_i\}_n)$ is the probability of finding a certain symbolic code of length n: $\{s_i\}_n = \{s_1, s_2, \ldots, s_n\}$.

The behavior of $Z_n(q)$ for large n is expected to be

$$Z_n(q) \sim \exp[P(q)n], \tag{2}$$

with $P(q) = K(q)(1-q)$, where $K(q)$ is the order-q Rényi entropy [8]. $P(q)$ is the metric analog of what is called in the mathematical literature the topological pressure [16].

B. The SH model

In the SH model, the 0's and 1's of the symbolic dynamics always correspond to the laminar and chaotic states, respectively. The basic idea is to factor the probability of a symbol sequence $p(\{s_i\}_n)$ by using the conditional probabilities $p(s_i \mid s_{i+1}s_{i+2}\cdots)$ of finding a sequence $s_{i+1}s_{i+2}\cdots$ after s_i. A given symbol sequence contains finite blocks of 0's and 1's. The characteristic time spent in the laminar region (0) is much longer than in the chaotic one. Therefore, the system will have a short-range memory on what happened in phase 1 if it is in phase 0 [17]. The SH model assumes that this is in fact a one-step memory, i.e., we can write $p(\cdots 10 \mid 00\cdots 01) = p(0 \mid 00\cdots 01)$. By writing the probability distribution $p(0 \mid 00\cdots 01)$ of laminar lengths as

$$p(0 \mid \underbrace{00\cdots 0}_{j-1} 1) = c_0 \exp[-W_0(j)], \tag{3}$$

one can introduce $W_0(j)$, a non-negative function, satisfying the conditions

$$W_0(1) = 0, \qquad c_0 = \left[\sum_{j=1}^{\infty} \exp[-W_0(j)]\right]^{-1} > 0. \tag{4}$$

In the remainder of the paper we assume that $W_0(j)$ is an increasing function for large j values, thus $p(0 \mid 0\cdots 01) \sim \exp[-W_0(j)]$ decreases with j.

Although this short-range-memory approximation introduces considerable errors when computing the $K(q)$ values (we found differences up to 50% with respect to the exact one), however, it yields the *exact* value for the critical point $q = q_c$ where a phase transition occurs, because it uses the correct scaling around $p(0\,|\,00\cdots00\cdots)$.

In the SH model the authors also suppose that the dynamics in the chaotic state can be represented by a Markov process. This means that the corresponding dynamical system has a one-step memory in phase 1. One can then write

$$p(1\,|\,\underbrace{11\cdots1}_{j-1}0) = p(1\,|\,1)^{j-1}p(1\,|\,0) \equiv b^{j-1}a, \tag{5}$$

where

$$a = p(1\,|\,0), \qquad b = p(1\,|\,1). \tag{6}$$

The constants a and b must then satisfy the normalization requirement

$$a + b = 1. \tag{7}$$

The SH theory proves that the (metric) topological pressure $P(q)$ can be obtained from the implicit equation

$$\lambda_{\rm SH}(P(q), q) = 1, \tag{8}$$

where $\lambda_{\rm SH}(P,q)$ is the largest eigenvalue of the matrix $T_{\rm SH}$:

$$T_{\rm SH} = \begin{pmatrix} 0 & c_0^q\Phi(P,q) \\ a^q e^{-P} & b^q e^{-P} \end{pmatrix}, \tag{9}$$

and

$$\Phi(P,q) \equiv \sum_{j=1}^{\infty} \exp\left[-Pj - qW_0(j)\right]. \tag{10}$$

One thus finds an equation for the topological pressure in the form

$$e^{-P}\left[b^q + (ac)^q\Phi(P,q)\right] = 1. \tag{11}$$

This equation allows the only solution $P(q) = 0$ for $q > q_c = 1$ if the PDLL exhibits a power-law decay [4]. We have not shown the derivation of these results here since a slightly more general version of the same formalism shall be worked out in detail in the next subsection.

C. An extended version of the SH formalism

As it is well known, nonanalyticies in the Rényi entropies are always related to the anomalous scaling of symbol sequence probabilities (or in terms of the dynamical system theory, of cylinder measures) [18]. The SH formalism correctly reflects the scaling of $\{00\ldots01\}$ symbol sequence probabilities: the orbits staying close to an eventually marginally stable fixed point in phase 0. This is why it succeeds in describing the behavior of $K(q)$ above the critical point q_c.

In order to include in this formalism possible nonanalyticies of $K(q)$ caused by an eventually anomalous scaling in phase 1, we do not assume the Markov property in this phase. We assume, however, that the system forgets its history at the transition from phase 0 to phase 1 too, and has an infinite-step memory inside each phase. We thus write the PDLL for phase 1 as

$$p(1\,|\,\underbrace{11\cdots1}_{j-1}0) = c_1 \exp\left[-W_1(j)\right], \tag{12}$$

where the function $W_1(j)$ and the constant c_1 satisfy conditions analogous to Eq. (4). These assumptions do not mean a considerable improvement from the point of view of the calculated $K(q)$ values, but this modified formalism is able to give the *exact* critical points of the $K(q)$ spectra in both phases.

In the extended formalism, we assume that the $p(\{s_i\}_n)$ probabilities can be factor as

$$p(s_1\cdots s_n) = p(s_1')p(s_1', s_2', j_1)p(s_2', s_3', j_2)\cdots p(s_k', s_1', j_k), \tag{13}$$

where

$$p(s,s',j) \equiv \begin{cases} p(s\,|\,\underbrace{s\cdots s}_{j-1}s') & \text{if } s \neq s' \\ 0 & \text{if } s = s' \end{cases} \tag{14}$$

and

$$s_1' = s_1, \qquad s_m' = s_{1+j_1+\cdots+j_{m-1}}, \qquad m = 2,\ldots,k. \tag{15}$$

In Eq. (13) a periodic boundary condition was assumed, which becomes irrelevant in the thermodynamic limit.

Let us consider the generating function ζ:

$$\zeta(P,q) = \sum_{n=1}^{\infty} e^{-Pn}Z_n(q). \tag{16}$$

For $n \gg 1$, one can write

$$\zeta(P,q) \sim \sum_{n=1}^{\infty} e^{-[P-P(q)]n} = \frac{e^{-[P-P(q)]}}{1-e^{-[P-P(q)]}} \tag{17}$$

as long as $P > P(q)$. Fixing q and varying P, $P = P(q)$ is the smallest zero of the function $1/\zeta$.

Using the factorization (13), the partition function can be written as

$$Z_n(q) = \sum_{k=1}^{\infty}\ \sum_{s_1,\ldots,s_k=0}^{1}\ \sum_{j_1,\ldots,j_k=1}^{\infty} \delta_{n,j_1+\cdots+j_m}[p(s_1)]^q[p(s_1,s_2,j_1)]^q[p(s_2,s_3,j_2)]^q\cdots[p(s_k,s_1,j_k)]^q, \tag{18}$$

where δ denotes the Kronecker symbol. The generating function then becomes

$$\zeta(P,q) = \sum_{k=1}^{\infty} \sum_{s_1,\ldots,s_k=0}^{1} [p(s_1)]^q \langle s_1 | T | s_2 \rangle \times \langle s_2 | T | s_3 \rangle \cdots \langle s_k | T | s_1 \rangle$$
$$= \sum_{k=1}^{\infty} \sum_{s_1=0}^{1} [p(s_1)]^q \langle s_1 | T^k | s_1 \rangle \quad (19)$$

where T is a 2×2 matrix with elements

$$\langle s | T | s' \rangle = \begin{cases} \sum_{j=1}^{\infty} e^{-Pj} [p(s,s',j)]^q & \text{if } s \neq s' \\ 0 & \text{if } s = s'. \end{cases} \quad (20)$$

Introducing the notation

$$\Psi(P,q) \equiv \sum_{j=1}^{\infty} \exp\left[-Pj - qW_1(j)\right], \quad (21)$$

matrix T reads as

$$T = \begin{pmatrix} 0 & c_0^q \Phi(P,q) \\ c_1^q \Psi(P,q) & 0 \end{pmatrix}. \quad (22)$$

Due to this simple form of the matrix, the ζ function becomes

$$\zeta(P,q) = \{[p(0)]^q + p[(1)]^q\} \frac{\langle 0 | T | 1 \rangle \langle 1 | T | 0 \rangle}{1 - \langle 0 | T | 1 \rangle \langle 1 | T | 0 \rangle}. \quad (23)$$

On the other hand, the largest eigenvalue of T is

$$\lambda(P(q),q) = (\langle 0 | T | 1 \rangle \langle 1 | T | 0 \rangle)^{1/2}. \quad (24)$$

From (24) and (23) we conclude that the smallest value $P = P(q)$, where the function ζ is diverging, is given by

$$\lambda(P(q),q) = 1. \quad (25)$$

Equation (25) can then be rewritten as

$$(c_0 c_1)^q \Psi(P(q),q) \Phi(P(q),q) = 1. \quad (26)$$

We note that matrix T is more general than the matrix appearing in the SH formalism. In the limit of uncorrelated chaotic states, the eigenvalue equation for this matrix becomes the same as in the SH theory. To see this, let us suppose that the bursts occur in a Markov process. Therefore, the probability of a symbol sequence with j turbulent symbols 1 can be written as in Eq. (5). By taking into consideration Eqs. (12) and (5)

$$c_1 = a, \quad (27)$$

$$W_1(j) = -(j-1) \ln b \quad (28)$$

follows, and Eq. (11) is recovered.

As results from Eq. (28), the property that chaotic symbols are uncorrelated is equivalent with the linearity of $W_1(j)$. A nonlinear form of $W_1(j)$ means that there are correlations between symbols, therefore we must not write the probabilities of such sequences as products of conditional probabilities. Also note that the deviations from linearity can happen in two obvious ways: $W_1(j)$ increases either slower or faster than linearly in j. These differences will be discussed in more detail in the following sections.

III. CRITERIA FOR PHASE TRANSITIONS—THE MAIN RESULTS

Our aim here is to study the conditions under which phase transitions between the chaotic phases (chaotic, regular, and anomalous chaos phase) arise. The criterion for the existence of a phase transition follows from the expression of $\Phi(P,q)$ and $\Psi(P,q)$ (10) and (21). We have nonanalycities in $P(q)$ [therefore in $K(q)$ too] whenever at least one of the sums appearing in Eq. (26) is diverging. A divergence arises when the exponent $Pj + qW(j)$ becomes negative for any j above some finite j_0 [where $W(j)$ is a shorthand notation for either $W_0(j)$ or $W_1(j)$]. As we shall see, this happens just in two situations corresponding to the limit cases when either

$$\beta(j) \to 0 \quad (29)$$

or

$$\beta(j) \to \infty, \quad (30)$$

where the shorthand notation

$$\beta(j) \equiv \frac{W(j)}{j} \quad (31)$$

has been introduced.

For the case when $W(j)$ increases slower than linearly, one observes that if $P(q)$ were negative for $q > 1$, the matrix element $c_0^q \Phi(P,q)$ [or $c_1^q \Psi(P,q)$] would be infinite, therefore the solution of Eq. (26) would not exists. Thus we conclude that $P(q) = 0$ for $q > 1$ and $q = q_c = 1$ is a CCP-RCP phase transition point [5,6,19]. This case is discussed in detail in Secs. IV and V.

For the second case when $W(j)$ is increasing faster than linearly, one also observes that if $P(q)$ were a positive finite quantity [$P(0) = \ln 2$] for $q < 0$, the matrix element $c_0^q \Phi(P,q)$ [or $c_1^q \Psi(P,q)$] would be infinite, therefore the solution of (26) would not exist. Then for $q < 0$ we get $P(q) = +\infty$, i.e., $q = 0$ is an ACP-CCP phase-transition point with an infinite jump.

The central result of this paper is to establish conditions for phase transitions in terms of the $W(j)$ functions which are explicitly measurable quantities from time series. These results can be summarized as follows.

(i) There is a phase-transition point at $q = 1$, and $P(q) = 0$ for $q > 1$ whenever $\beta(j) \to 0$ for at least one of the $W(j)$ functions (or symbols).

(ii) When $\beta(j) \to \alpha$ (α is a nonzero positive finite number) for all the $W(j)$ functions, the whole entropy spectrum is smooth, and no phase transition occurs.

(iii) There is another phase transition at $q = 0$, and

$P(q) = +\infty$ for $q < 0$ whenever $\beta(j) \to \infty$ for at least one of the $W(j)$ functions.

Moreover, we prove that both in the regular (RCP) and anomalous (ACP) phases the topological pressure converges to the corresponding limiting values 0 and ∞, respectively, in a way that is determined solely by the asymptotics of the PDLL (see Sec. IV). In particular, defining $P_k(q)$ as the solution of the eigenvalue equation $\lambda(P_k(q), q) = 1$ for the matrix T, in which the first k terms are kept in the divergent sums (Ψ,Φ, or both), we find for the limiting behavior $P_k(q) \to P(q)$ that

$$P_k(q) \sim -q\beta(k), \qquad k \gg 1. \tag{32}$$

This formula also shows that properties (29) and (30) imply the existence of an RCP and ACP, respectively.

In order to classify *intermittent* systems (where the PDLL takes on relatively large values for short and intermediate lengths) from the point of view of the generated chaotical phases, we are interested in the probability distribution of the *laminar* lengths, i.e., in those $W(j)$ functions which describe *laminar* symbols. Symbol 0 is, by definition, a laminar one, but in certain cases symbol 1 might also appear in long laminar blocks. As we shall see in Secs. V–VII, not only case (i) (as it is widely believed), but also (ii) and (iii) can occur for *laminar* symbols. Therefore, one can distinguish three types of intermittencies: *classical intermittency*, when $\beta(j) \to 0$; *borderline intermittency*, when $\beta(j) \to \alpha$; and *anomalous intermittency*, when $\beta(j) \to \infty$, for a laminar symbol. In order to illustrate more properly the classification, in Secs. V–VII we direct our attention to the class of one-dimensional maps which show fully developed chaos.

IV. ASYMPTOTIC FORMS IN THE REGULAR AND ANOMALOUS PHASES

In order to characterize the approach to the exact value of $P(q)$ we shall use the following technique: cut off the sum (10) at a finite value $k \gg 1$, i.e., consider

$$\Phi_k(P,q) = \sum_{j=1}^{k} \exp\left[-P(q)j - qW_0(j)\right]. \tag{33}$$

Let $P_k(q)$ the solution of the equation

$$\lambda_k(P_k(q), q) = 1, \quad q < 1, \; k \gg 1, \tag{34}$$

where λ_k is the largest eigenvalue of the truncated matrix T_k constructed as in (22) with Φ_k instead of Φ. The explicit form of (34) is

$$\lambda_k(P_k, q) \equiv (c_0 c_1)^q \Psi(P_k, q)\Phi_k(P_k, q) = 1. \tag{35}$$

A. RCP situation [$q > 1$ and $\beta_0(k) \to 0$]

The following inequality is valid:

$$\Phi_k(P,q) = \sum_{j=1}^{k} \exp\left[-Pj - qW_0(j)\right] > \frac{1 - e^{-Pk}}{1 - e^{-P}} e^{-qW_0(k) - P}. \tag{36}$$

Let us take a special value of P, namely $P = -q\beta_0(k)$. Then we obtain for $k \gg 1$

$$\Phi_k(-q\beta_0(k), q) > \frac{e^{q\beta_0(k)}}{e^{q\beta_0(k)} - 1}\left(1 - e^{-qW_0(k)}\right) \gg 1. \tag{37}$$

We assume that the behavior of symbol 1 is normal, i.e., $\beta_1(j) \to \alpha$ $(0 < \alpha < \infty)$. Thus

$$\Psi_k(-q\beta_0(k), q) = \sum_{j=1}^{\infty} \exp\{-qj[\beta_1(j) - \beta_0(k)]\} \sim \sum_{j=1}^{\infty} \exp\left(-qj\alpha\right) \tag{38}$$

is a finite number. One also observes that $\lambda_k(P,q)$ is a decreasing function of P, i.e.,

$$\frac{\partial \lambda_k}{\partial P} < 0. \tag{39}$$

From (37) it follows that for $k \gg 1$

$$\lambda_k(-q\beta_0(k), q) \gg 1. \tag{40}$$

Now, due to (35), (39), and (40) we get

$$-q\beta_0(k) < P_k < 0. \tag{41}$$

Next, we show that the asymptotic behavior (32) holds. Equation (35) is equivalent to

$$\Phi_k(P_k, q) - C = 0, \tag{42}$$

where $C = c_0^{-q} c_1^{-q} \Psi^{-1} > 1$ by taking into consideration (38) and the expression of c_0^{-q} and c_1^{-q} (4).

One can assume for $k \gg 1$ that every term in Φ_k, except the first ($j = 1$), can be majorated by the last one (the kth) as

$$\exp\{-k[P_k + q\beta_0(k)]\} > \exp\{-j[P_k + q\beta_0(j)]\}, \qquad j = 2, \ldots, k-1. \tag{43}$$

Thus, from (42) it follows that

$$(k-1)\exp\{-k[P_k + q\beta_0(k)]\} + e^{-P_k} - C > 0. \tag{44}$$

This results in

$$P_k < -q\beta_0(k) + \frac{\ln(k-1)}{k}. \tag{45}$$

From (41) and (45) we conclude that for $k \gg 1$

$$P_k(q) \sim -q\beta_0(k), \tag{46}$$

provided $W_0(k) > \ln k$. Otherwise condition (4) is violated.

B. ACP situation [$q < 0$ and $\beta_0(k) \to \infty$]

As mentioned in Sec. III, P_k must diverge in this case. Therefore

$$\Psi_k(P_k, q) = \sum_{j=1}^{\infty} \exp\{-j[P_k + q\beta_1(j)]\}$$
$$\approx \sum_{j=1}^{\infty} \exp(-jP_k) \approx e^{-P_k} \tag{47}$$

since P_k is much larger than $q\beta_1(j) \approx q\alpha$ for $k \gg 1$. Then, Eq. (35) reads

$$\sum_{j=1}^{k} \exp\left[-(j+1)\left(P_k + q\frac{W_0(j)}{j+1}\right)\right] - (c_0c_1)^{-q} = 0. \tag{48}$$

The following inequalities hold:

$$\exp\left[-(k+1)\left(P_k + q\frac{W_0(k)}{k+1}\right)\right] < (c_0c_1)^{-q}, \tag{49}$$

$$\exp\left[-(k+1)\left(P_k + q\frac{W_0(k)}{k+1}\right)\right]$$
$$> \exp\left[-(j+1)\left(P_k + q\frac{W_0(j)}{j+1}\right)\right], \quad j = 1, \ldots, k. \tag{50}$$

This means

$$-q\frac{W_0(k)}{k+1} - q\frac{\ln(c_0c_1)}{k+1} < P_k < -q\frac{W_0(k)}{k+1} - q\frac{\ln(c_0c_1)}{k+1} + \frac{\ln(k+1)}{k+1}. \tag{51}$$

The latter formula is equivalent to saying that

$$P_k(q) \sim -q\beta_0(k) \tag{52}$$

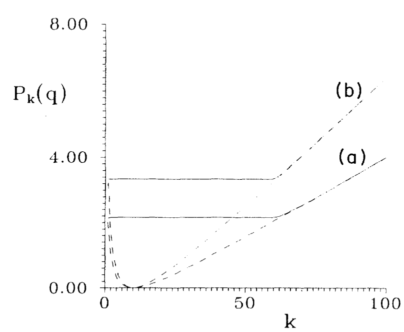

FIG. 1. $P_k(q)$ (solid line) and $-q\beta_0(k)$ (dashed line) vs the truncating index k for a Gaussian intermittent system (see Sec. VII) [$\beta_0(k) \sim (k-L)^2/k$] for $L = 10$ and $p(1\,|\,0) = 0.25$ at (a) $q = -0.05$ and (b) $q = -0.08$.

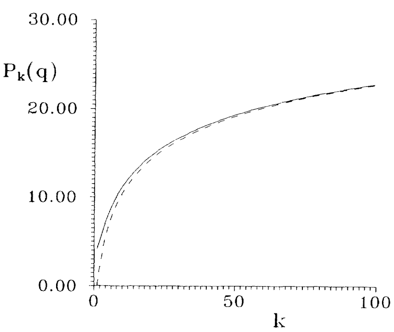

FIG. 2. $P_k(q)$ (solid line) and $-q\beta_0(k)$ (dashed line) vs the truncating index k for $\beta_0(k) = \ln k$ and $p(1 \mid 0) = 0.25$ at $q = -5$.

for $k \to \infty$. From (52) $K_k(q) \sim q\beta_0(k)/(q-1)$ follows. Figures 1 and 2 serve as numerical evidence for the scaling formula (52), solving (34) by using the tangent method for two different choices of β_0: $\beta_0(k) \sim (k-L)^2/k$ and $\beta_0(k) \sim \ln k$, respectively. Note that $P_k(q)$ is very close to $-qW_0(k)/(k+1)$ even at lower k values (about 100).

The anomalous scaling of the PDLL, both in the RCP and in the ACP, has consequences for the scaling of the partition sums, too. Analyzing the formalism presented in Sec. II C, one can observe that for large enough n and k, the truncating index k can be replaced by n. Therefore, we conjecture that for large enough n the partition function (1) scales as

$$Z_n(q) \sim \exp[-qW(n)]. \tag{53}$$

This is a generalization of the form suggested in Ref. [9], where $\ln Z_n(q) \sim G^n(q)$ was assumed in the ACP with $G(q) > 1$ being some finite function of q.

V. CLASSICAL INTERMITTENCY

The $W(j)$ function for a laminar state is determined, on one hand, by the specific form of the map around the fixed point and, on the other hand, by the reinjecting branch. Therefore, classical intermittency can be obtained either by a marginally stable fixed point and uniform reinjection [Fig. 3(a)] or by an unstable fixed point and a suitable choosen reinjecting branch with a strong singularity [Fig. 4(a)] [18]. Althought only the first case is intermittent in the sense of Pomeau-Manneville [11,12], as we can observe in Figs. 3(b) and 4(b), both cases show qualitatively the same intermittent signal. In our sense, not only type-II and -III intermittency [where $W_0(j)$ is a power-law function [1,4]] belong to classical intermittency, but also some intermittent maps *without a marginally stable fixed point* but with a strong singular reinjection (see Appendix, case I), which exhibit a phase transition at $q_c = 1$ [18]. In order to understand this phenomenon, let us say a few words about it in terms of the dynamical systems theory. A phase transition occurs in the Rényi entropies $K(q)$ whenever there exists a cylinder probability with anomalous scaling. In the sec-

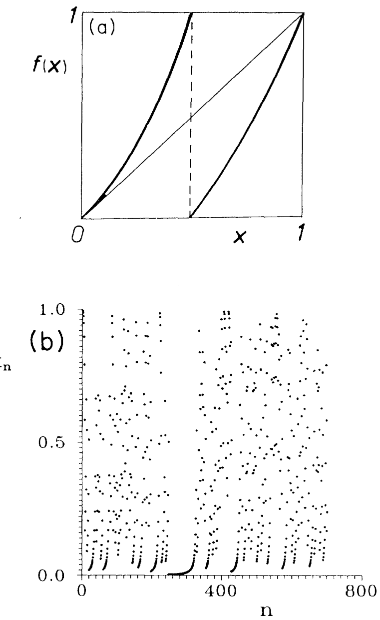

FIG. 3. (a) Map with a marginally stable fixed point and uniform reinjection and (b) the generated intermittent signal after omitting the transients.

ond case the cylinder length scaling is exponential, but the invariant density is singular around the fixed point. This corresponds to the existence of an unstable fixed point with singular reinjection. The same situation was pointed out by Bene and Szépfalusy; see Ref. [18].

Up to now, we have focused our attention on the possible effects of the intermittent phase 0 on the $K(q)$ spec-

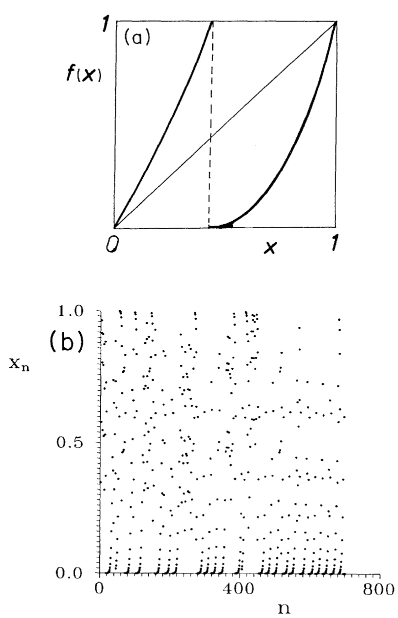

FIG. 4. (a) Map with unstable fixed point and strongly singular reinjection [Appendix, case I] and (b) the signal generated. One can observe intermittent signal, although no marginal fixed point exists.

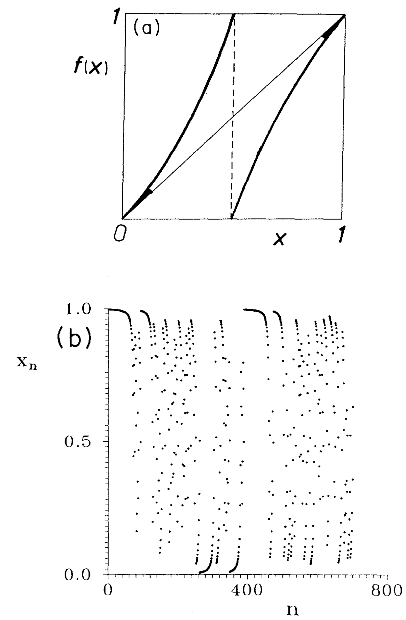

FIG. 5. (a) Map with two marginal fixed points. The encoded signal consists of two kinds of laminar states. (b) The bursts observed in the generated signal are in fact short laminar states.

tra. However, in specific 1D maps, a possible anomalous scaling in phase 1 could also cause additional nonanalyticies in $K(q)$. Our modified formalism takes into consideration these phenomena, too. According to this, there are two possibilities [here $\beta_i(j)$ stands for $W_i(j)/j$, $i \in \{0,1\}$].

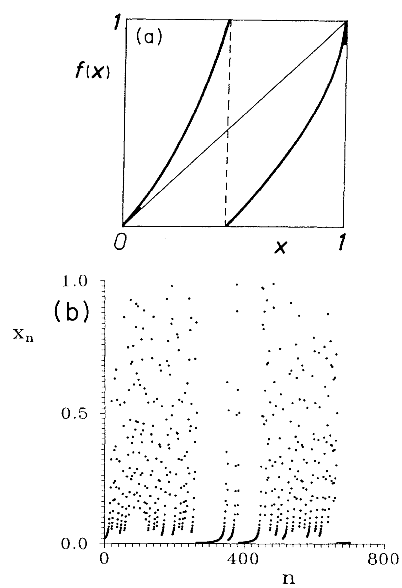

FIG. 6. (a) Lorenz type map, which shows the double-phase transition (CCP-RCP and ACP-CCP) and (b) the generated signal. Comparing with Fig. 3(b), the effect of the superunstable fixed point shows up in the low point density in the upper half of the figure.

(a) $\beta_0(j) \to 0$ and $\beta_1(j) \to 0$ and $(1 \in \mathcal{E}_1)$. This is a very interesting physical situation because in the signal we observe just two kinds of laminar states (see Fig. 5). Chaos, however, still exists since the two laminar states alternate with chaotically varying lengths. It can be observed that we then have just a single phase-transition point at $q = 1$ and $P(q) = 0$ for $q > 1$ (RCP). Following the same arguments as in Sec. IV, it can be shown that the convergence of $P_k(q)$ towards 0 is governed by the minimum of $\beta(k)$, i.e., by

$$\beta(k) = \min\{\beta_0(k), \beta_1(k)\}.$$

(b) Another interesting case is when $\beta_0(j) \to 0$ and $\beta_1(j) \to \infty$. Here we have a double phase transition which is due to the coexistence of a marginally stable and a superunstable fixed point. A good example is a Lorenz-type map investigated in Ref. [22]; see Figs. 6(a) and 6(b).

VI. BORDERLINE INTERMITTENCY

As mentioned in the Introduction and Sec. III, our definition of intermittency allows cases when $\beta_0(j)$ does not go asymptotically to zero, provided the PDLL takes on nonnegligible values for short and intermediate lengths. Here we consider cases when $\beta_0(j) \to$ const, which implies the absence of any phase transition in the $K(q)$ spectrum as follows from the previous discussion.

The name "borderline" is choosen because the set of $\beta(j)$ functions belonging to this case (the asymptotically constant ones) is of measure zero in the set of the possible $\beta(j)$ functions, and provides a "border" between the classes characterized by different dynamical phase transitions, that is, by asymptotic behaviors (29) and (30).

Intermittency is not necessary related to the existence of a marginally stable fixed point [18]. In this case we might observe intermittent behavior when α, the limiting constant of $\beta_0(j)$, is much less than unity.

As a more interesting situation, let us suppose that the system posesses a marginal fixed point. How is it possible to have an exponential scaling of the PDLL $p(0 \mid 0 \cdots 01) \sim e^{-\alpha j}$ or of the natural measure around such a fixed point? This could happen because $W_0(j)$ characterizes the asymptotic properties of the signal which could reflect the result of the competition between the marginal fixed point and other eventually existing singularities. To ensure the exponential decrease of the PDLL, one needs a special form for the reinjecting branch with a strong singular piece which cannot be expressed in a polynomial form. To see this, let us consider the simple case when x_c is a singular point of the reinjecting branch, mapped in one step into the intermittent fixed point (Fig. 7). In the Appendix we calculated $F_1(y)$, the inverse of the reinjecting branch around this singular point in order to have a given dependence for $\beta_0(j) = W_0(j)/j$ if $j \gg 1$. In the present case $\beta_0(j) = \gamma j^{\tau-1}$, where γ and τ are positive constants and $\tau = 1$ (Appendix, case II). It can be observed that $F_1(y)$ near $y = 0$ is fairly singular, depending on y via $\exp(-\delta\frac{1}{y})$, where $\delta = \gamma[f''(0)]^{-\tau}$, provided the origin is a marginally stable fixed point $[f'(0) = 1]$.

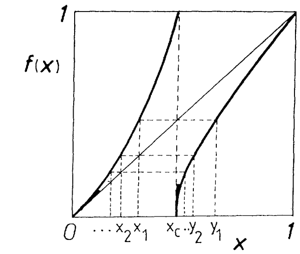

FIG. 7. Map with strong singular reinjection. The singular point x_c is mapped in one step onto the fixed point from the origin. The points x_j are the preimages of x_c, taken by F_0 and y_j, the image of x_j, taken by the F_1 branch. One can observe that the probability of an intermittent orbit of length j is proportional to the distance $y_j - y_{j+1}$ for $j \gg 1$.

We are emphasizing that the strong singularity of $F_1(y)$ needed for the exponential decay of the PDLL (for $j \gg 1$) holds in a very small neighborhood of $y = 0$ (or $x = x_c$) only. The intermittent behavior around the marginally stable fixed point $(x = 0)$ is "screened" in a very small vicinity, say in the interval $[0, \epsilon)$ $(\epsilon \ll 1)$. The shape of the reinjecting branch further than ϵ $(y \geq \epsilon)$ is assumed to be locally linear. Therefore, the narrow channel formed by $F_0(y)$ and the first diagonal has a width of order ϵ in the region $x \geq \epsilon$ which is not screened and is responsible for the intermittent behavior observed.

VII. ANOMALOUS INTERMITTENCY

As we have seen in the preceding section, intermittency is not necessarelly related to the condensed (RCP) phase. In the case of borderline intermittency, the PDLL is characterized by an exponential decay. However, it is possible to find a faster than exponential PDLL too. As follows from Secs. III and IV, this introduces an anomalous chaos phase for $q < 0$, thus a phase transition occurs at $q = 0$ with an infinite jump.

Here the considerations of the last paragraph of Sec. VI also hold, namely the intermittent behavior is generated by the "nonscreened" part, $x \geq \epsilon$ of the branch 0.

As an example for this type of intermittency, let us consider the map sketched in Fig. 7 with the claim that $\beta_0(j) \to \infty$. With the notations of the Appendix, let us consider for, e.g., $\beta_0(j) = \gamma j^{-\tau}$, $\gamma > 0$, $\tau > 1$, and the map f having the first diagonal as tangent in the origin. A strong dependence of $F_1(y)$ on y in the form of $(\frac{1}{y})^{1-\tau} e^{-\delta(1/y)^\tau}$ follows (see Appendix, case II) which is stronger than in the borderline case (when $\tau = 1$).

It is interesting to consider cases where the PDLL exhibits a definite maximum at some laminar length $L > 1$. In particular, we shall assume that the probability distribution of laminar lenghts is approximated by a normal distribution. For the sake of simplicity, we shall refer to this case as a Gaussian intermittent system (GIS). Thus for a GIS $W(j) \sim (j - L)^2$, where L is the most probable laminar length. It is obvious that $\beta_0(j) \to \infty$, and ACP exists for $q < 0$. Figure 1 shows the function $P_k(q)$ vs k for two different q values at a fixed L. We can observe an interesting feature which is typical for anomalous intermittency (so for GIS): the existence of a k-independent region for $P_k(q)$ (horizontal line) and another one which characterizes the scaling (32) at large k. The transition between the two regions takes place smoothly, but in a very thin interval. Rigourously speaking, the k independent region is not exactly k independent, but the dependence shows up just in the ninth and tenth digits. This can be explained by the fact that the new term, which appears in Eq. (48) by increasing k, is very small: the part $\exp[-P_k(k+1)]$ ($P_k > 0$) decreases more rapidly than the exponential $\exp[-q(k-L)^2]$ increases for $L < k < 60$ (in the special case of Fig. 1). For k greater than a certain value, $\exp[-q(k-L)^2]$ becomes more important and the solutions $\exp(-P_k)$ become smaller. Numerical simulations also show that the q dependence of $P_k(q)$ in the horizontal region is almost linear [not just for large k, as seen in (32)]. With the increase of the most probable laminar lenght L the horizontal region becomes larger.

VIII. CONCLUDING REMARKS

The criterion given in this paper provides us with the ability to decide whether there exist phase transitions in the $K(q)$ spectrum by studying just the intermittent signal and deducing its PDLL without constructing explicitly the dynamics of the system. We tried to point out that intermittency is a more complex phenomenon than it is usually considered to be, even in simple 1D maps. The point where this complexity comes from is the definition of the intermittent behavior via an "eyeballing" of the signal. This cannot distinguish usual intermittent cases (in the Pomeau-Manneville sense) from other ones and does not reflect the type of the time correlation function. Defining the intermittency as we propose in this paper gains a somewhat closer image concerning the possible phase transitions in the spectrum of the Rényi entropies. It is worth emphasizing again that the existence of a marginally stable fixed point is neither a necessary (Sec. V) nor a sufficient condition (Secs. VI and VII) for intermittency in contrast to the general belief.

If we are using more symbols to encode the signal, the main results presented in Sec. III remain valid because, in spite of the fact that the matrix T becomes more complicated (for N symbols T is an $N\times N$ matrix), it has entries of type $\Phi_i(P,q) \equiv \sum_{j=1}^{\infty}\exp[-Pj - qW_i(j)]$ (where i stands for the ith symbol), with zeros on the diagonal; therefore the criterion above still holds. Even the scaling formula (32) remains valid with $\beta(k)$ now being the slowest behaving function between the symbols satisfying (29) or (30) in the regular chaos phase and the anomalous chaos phase, respectively.

Phase transitions connected with type II and III are well described by the SH model, but type I needs a more careful treatment because of the existing narrow channel rather than a marginally stable fixed point. This introduces qualitative differences, e.g., in the value of the critical point q_c, which might become in this case less than 1 [6,20,21], or in the scaling behavior near the critical point which is characterized by exponents depending on the value of q_c [7]. The only change needed in the SH formalism to characterize type-I cases is the introduction of three symbols: two for laminar states $-1, 0 \in \mathcal{E}_1$ and one for chaotic ones $1 \in \mathcal{E}_2$. In this case, T is a 3×3 matrix for which Eq. (25) reads

$$e^{-P(q)} = 1, \tag{54}$$

$$(c_0 c_1)^q \Phi(P,q)\Psi(P,q) = 1, \tag{55}$$

where c_0 is the same as defined in (4), but c_1 satisfies the normalization condition $(c_1 + c_1')\sum_{j=1}^{\infty}\exp[-W_1(j)] = 1$, with $c_1 = p(1\,|\,0)$ and $c_1' = p(1\,|\,-1)$. Note that Eq. (55) is the same as (26), but there is an extra solution given by Eq. (54) in the form of $P(q) = 0$. Due to the fact that now $c_1\sum_{j=1}^{\infty}\exp[-W_1(j)] < 1$, Eq. (55) can be fulfilled by the solution $P(q) = 0$ at $q = q_c < 1$. Physically, this is explained by the existence of a repellor in the phase space with Hausdorff dimension q_c and c_1' characterizing the escape from this repellor (see also Ref. [20]).

The SH or the extended model takes into consideration exactly only the scaling behavior around the main fixed points $00\cdots0\cdots$ and $11\cdots1\cdots$. The other periodic points are taken into consideration just approximately, which can cause serious errors. We believe that, by taking into consideration more unstable periodic points, the theory gains an improvement such that it can be applied to nonintermittent cases too. This needs to give the $W_i(j)$ functions for a few primitive cycles as $i \in \{0, 1, \overline{01}, \overline{001}, \overline{101}, \ldots\}$ and the corresponding transition probabilities. We expect that the number of cycles needed is small such that the eigenvalue equation for the matrix T can be handled relatively easy. A detailed analysis of these problems would be interesting.

ACKNOWLEDGMENTS

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APPENDIX

As we have seen in Sec. V, the PDLL is determined by the map around the fixed point and, on the other hand, by the reinjecting branch. Let us consider a map f with the presentation functions F_0 and F_1 denoting the two inverse branches of f. Suppose that the behavior around the fixed point is governed by (the given function) F_0. Here we solve the problem of what the form of the reinjecting branch F_1 around the singular point x_c, which is mapped in one step onto the origin [see Figs. 4(a) and 7], should be in order to obtain a probability distribution for the laminar lengths characterized by a given $W_0(j)$ function. First, we derive the general formula and then work out some particular cases in order to see the screening effect of the singular point x_c on the fixed point.

Let us define the points x_j as the preimages of x_c taken by F_0,

$$x_{j+1} = F_0(x_j), \quad x_0 = x_c, \tag{A1}$$

and y_j as their image taken on the F_1 branch,

$$y_{j+1} = F_1(x_j) \tag{A2}$$

[see Figs. 4(a) and 7]. If $j \gg 1$, one can write

$$p(0|\underbrace{0\cdots0}_{j-1}1) = \kappa(y_j - y_{j+1}), \tag{A3}$$

where κ is a constant. Using the relation $f'(y_{j+1}) = [f(y_j) - f(y_{j+1})]/(y_j - y_{j+1})$ and Eqs. (A1) and (A2),

$$y_j - y_{j+1} = F_1'(x_j)(f(x_j) - x_j). \tag{A4}$$

Introducing the step function

$$\mathcal{J}(x) = j \quad \text{if } x \in (x_j, x_{j-1}], \tag{A5}$$

and using (3), Eq. (A3) becomes

$$c_0 e^{-(\beta_0 \circ \mathcal{J})(y)\mathcal{J}(y)} = \kappa F_1'(y)(f(y) - y), \tag{A6}$$

where $y \equiv x_j$. This is equivalent to

$$F_1'(y) = \kappa' \frac{e^{-(\beta_0 \circ \mathcal{J})(y)\mathcal{J}(y)}}{f(y) - y}, \quad 0 < y \ll 1, \tag{A7}$$

with $\kappa' = c_0/\kappa$. For given $\beta_0(j)$ and $f(x)$ $(0 < x \leq x_c)$ functions Eq. (A7) can be solved by direct integration.

In the following we restrict ourselves to a class of $\beta_0(j)$ functions of the form

$$\beta_0(j) = \gamma j^{\tau-1}, \quad \gamma, \tau > 0. \tag{A8}$$

As follows from (A7), the form of the reinjecting branch $F_1(y)$ around $y = 0$ is determined solely by the slope of the other branch taken at the origin. Therefore the cases $f'(0) > 1$ and $f'(0) = 1$ have to be treated separately.

Case I: $f'(0) > 1$ [see Fig. 4(a)]. Around the origin

$$f(y) - y \approx [f'(0) - 1]y \equiv ay. \tag{A9}$$

Consequently, for the function $\mathcal{J}(y)$ we obtain

$$\mathcal{J}(y) \approx \frac{1}{a} \ln \frac{1}{y}. \tag{A10}$$

Inserting (A8)–(A10) in (A7), we get, after integration,

$$F_1(y) = x_c + \kappa' \int_{\ln 1/y}^{\infty} e^{-\omega z^\tau} dz, \tag{A11}$$

where $\omega = \gamma a^{-\tau}$. The expression above can be transformed into

$$F_1(y) = x_c + \kappa' \frac{1}{\tau \omega^{1/\tau}} \int_{v_0}^{\infty} e^{-v} v^s ds, \tag{A12}$$

where $s = \frac{1}{\tau} - 1$ and $v_0 = \omega(\ln \frac{1}{y})^\tau \to \infty$ as $y \to 0$. The integral can be expressed by means of the Whittaker functions $\mathcal{W}_{\lambda,\mu}$ [23] as

$$\int_{v_0}^{\infty} \frac{e^{-v}}{v^{-s}} dv = v_0^{s/2} e^{-v_0/2} \mathcal{W}_{s/2,1+s/2}(v_0). \tag{A13}$$

Using the asymptotic expression of $\mathcal{W}_{s/2,1+s/2}(v_0)$ ($v_0 \gg 1$) [23] finally results in

$$F_1(y) = x_c + C_1 \left(\ln \frac{1}{y}\right)^{1-\tau} e^{-\omega(\ln 1/y)^\tau}. \tag{A14}$$

Obviously, we have an RCP whenever $0 < \tau < 1$; see (A8). Therefore, the map f will generate qualitatively the same signal as a model for type-II or -III intermittency [see Fig. 3(b) and 4(b)]. This is accompanied by a CCP-RCP transition in the $K(q)$ spectrum at $q = q_c = 1$, in spite of the absence of a marginally stable fixed point (see Sec. V). If $\tau = 1$, then $F_1(y) = x_c + C_1 y^\omega$ is a power-law function and no phase transitions occur in the $K(q)$ spectrum. Similarly, the case of $\tau > 1$ does not correspond to an intermittent system either.

Case II: $f'(0) = 1$ (see Fig. 7). Around the origin

$$f(y) - y \approx \frac{1}{2} f''(0) y^2 \equiv by^2. \tag{A15}$$

This yields, for $\mathcal{J}$,

$$\mathcal{J}(y) \approx \frac{1}{by}. \tag{A16}$$

So, Eq. (A7) reads

$$F_1'(y) = \kappa' \frac{1}{y^2} e^{-\delta y^{-\tau}},$$

where $\delta = \gamma b^{-\tau}$. After evaluating the integral above, we obtain

$$F_1(y) = x_c + \kappa' \frac{1}{\tau \delta^{1/\tau}} \int_{w_0}^{\infty} e^{-v} v^s ds \tag{A17}$$

with $w_0 = \delta y^{-\tau} \to \infty$ as $y \to 0$. Using the same approximations as in case I, finally we get

$$F_1(y) = x_c + C_2 \left(\frac{1}{y}\right)^{1-\tau} e^{-\delta(1/y)^\tau}. \tag{A18}$$

It is evident that for $\tau = 1$ $F_1(y)$ depends on y via $\exp(-\delta \frac{1}{y})$, which shows the strong singularity needed to cancel the phase transition. If $\tau > 1$, then the reinjecting branch around $y = 0$ takes a more singular form (A7) as in the borderline case ($\tau = 1$), and in the $K(q)$ spectrum we observe the CCP-ACP transition. For $\tau < 1$ the reinjecting branch enhances intermittency and does not remove the RCP phase.

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